



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 51

Futures - 1

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BLACK SCHOLES EQUATION

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf = 0 \quad (1)$$

BOUNDARY CONDITIONS

For a European call option :

$$f(S_T, T) = c(S_T, T) = \max(S_T - K, 0)$$

For a European put option :

$$f(S_T, T) = p(S_T, T) = \max(K - S_T, 0)$$



BS SOLUTIONS FOR CALLS & PUTS

$$c = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2)$$

$$p = Ke^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

OPTIONS ON DIVIDEND PAYING STOCK

$$c = (S_0 - D_0) \Phi(d_1) - Ke^{-rT} \Phi(d_2)$$

$$p = Ke^{-rT} \Phi(-d_2) - (S_0 - D_0) \Phi(-d_1)$$

$$d_1 = \frac{\ln((S_0 - D_0)/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln((S_0 - D_0)/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

OPTIONS ON INDICES

$$c = S_0 e^{-qT} \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

$$p = K e^{-rT} \Phi(-d_2) - S_0 e^{-qT} \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

OPTIONS ON CURRENCY

$$c = S_0 e^{-r_f T} \Phi(d_1) - K e^{-r T} \Phi(d_2)$$

$$p = K e^{-r T} \Phi(-d_2) - S_0 e^{-r_f T} \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r - r_f + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - r_f - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

EXAMPLE 1

- A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is 38. What is the probability that a European call option on the stock with an exercise price of 40 and a maturity date in 6 months will be exercised?

The required probability is the probability of the stock price being above \$40 in six months time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim N \left(\ln 38 + \left(0.16 - \frac{0.35^2}{2} \right) 0.5, 0.35^2 \times 0.5 \right) \text{ i.e.,}$$

$$\ln S_T \sim N(3.687, 0.247^2)$$

Since $\ln 40 = 3.689$, the required z value is $z = \frac{3.689 - 3.687}{0.247} = 0.008$

We want $P(Z > z = 0.008) = 1 - P(Z < z = 0.008) = 1 - \Phi(0.008)$

From normal distribution tables $\Phi(0.008) = 0.5032$ so that the required probability is 0.4968

THREE IMPORTANT RESULTS ON BLACK SCHOLES MODEL

A. Prob of exercise of call option in risk neutral world
 $= P(S_T > K) = P(\ln S_T > \ln K) = \Phi(d_2)$

B. $\Delta = \frac{\partial c}{\partial S} = \Phi(d_1)$

C. Conditional Expectation of Stock Price Subject to Call Exercise
 $= E[S_T | (S_T \geq K)] = S_0 e^{rT} \Phi(d_1)$

INTERPRETATION OF BS SOLUTION

$$c = e^{-rT} E_Q [f(S_T)] = e^{-rT} E_Q [\max(S_T - K, 0)]$$

In the Black Scholes model:

$$c = e^{-rT} [e^{rT} S_0 \Phi(d_1) - K \Phi(d_2)]$$

First component of payoff at T: $C_T^1 = S_T$

***Second* component of payoff at T: $C_T^2 = -K$**

First component of payoff : $C_T^1 = S_T$

However, it will be paid if the option finishes in the money

Hence, we can write this component as a contingent payoff

$$C_T^1 = \begin{cases} +S_T & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

Now, expected value of C_T^1

$$\begin{aligned} E(C_T^1) &= E[S_T | (S_T \geq K)] + 0 \cdot P(S_T < K) = E[S_T | (S_T \geq K)] \\ &= e^{rT} S_0 \Phi(d_1) \text{ so that } S_0 \Phi(d_1) = \text{PV of } E(C_T^1) \end{aligned}$$

**Second component of the payoff : $C_T^2 = -K$
However, it will be paid if the option
finishes in the money**

**Hence, we can write this component
as a contingent payoff**

$$C_T^2 = \begin{cases} -K & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

Now, expected value of C_T^2

$$\begin{aligned} E(C_T^2) &= -K \cdot P(S_T \geq K) + 0 \cdot P(S_T < K) \\ &= -K \cdot P(S_T \geq K) = -K \cdot \Phi(d_2) \end{aligned}$$

so that $-e^{-rT} K \cdot \Phi(d_2) = \text{PV of } E(C_T^2)$

$$\begin{aligned}
 \text{Hence } c &= e^{-rT} \left[e^{rT} S_0 \Phi(d_1) - K \Phi(d_2) \right] \\
 &= \text{PV of } E(C_T^1) - E(C_T^2) \\
 &= \text{PV of } E(C_T^1 - C_T^2) \\
 &= \text{PV of } E(\text{net cash flow from option})
 \end{aligned}$$

ALTERNATIVE INTERPRETATION OF BS SOLUTION

- Let us assume that you write a call option. Then, you create an obligation to honour the call, if the option holder decides to exercise the option.
- However, you can cover this risk exposure and maintain your position riskless by taking a long position in Δ units of stock.

$$\Delta = \frac{\partial c}{\partial S} = \Phi(d_1)$$

- In other words, by creating a portfolio of a short call and long Δ units of stock you create a riskless portfolio and hence, your overall risk position is unaffected.
- However, buying Δ units of stock entails a cash outflow of $\Delta S_0 = S_0 \Phi(d_1)$. This cashflow occurs at $t=0$.
- Against this, you have the possibility of receiving the exercise price K .

- But there are two things here:
- (i) K will be received on option maturity. Hence, it is the present value of K i.e. Ke^{-rT} that is relevant;
- (ii) The probability of receiving K is equal to the probability that the option will be exercised $=\Phi(d_2)$.
- Hence, weighted cash inflow on writing the call option is $Ke^{-rT}\Phi(d_2)$.
- Thus, net value of call to writer: $-c = Ke^{-rT}\Phi(d_2) - S_0\Phi(d_1)$

BS FORMULA: RISK NEUTRAL DERIVATION

$$\begin{aligned}c &= e^{-rT} \mathbf{E}_Q \left[\max(S_T - K, 0) \right] \\&= e^{-rT} \mathbf{E}_Q \left[(S_T - K) | S_T > K \right] \\&= e^{-rT} \left\{ \mathbf{E}_Q \left[S_T | S_T > K \right] - \mathbf{E}_Q \left[K | S_T > K \right] \right\} \\&= e^{-rT} \left\{ \mathbf{E}_Q \left[S_T | S_T > K \right] - K P_Q(S_T > K) \right\} \\&= e^{-rT} \left[e^{rT} S_0 \Phi(d_1) - K \Phi(d_2) \right] \\&= S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)\end{aligned}$$



FUTURES



WHAT ARE FUTURES?

- Agreement to buy or sell an asset for a certain price at a certain time.
- Similar to forward contracts.
- Linear payoff function $\Pi_T = S_T - F_0$.
- Futures contracts traded on an exchange.
- A pre-requisite of exchange tradability is that futures contracts must necessarily be standardized.
- Secondly, mechanism needs to be in place for elimination of default risk.



TERMINOLOGY

- **Lot size** is the minimum quantity specified in the futures contract.
- **Contract value = Futures Price * Lot Size.**
- **Expiry** is the last date up to which one can hold the futures.
- **Margin**: To enter into a futures agreement one has to deposit a **margin** amount, which is calculated as a certain % of the contract value.



SETTLEMENT: PHYSICAL VS CASH

- If a contract is cash settled, when the contract expires, margin account will be marked-to market with the spot price at settlement for P&L on the final day of the contract.
- In case of cash settlement, there is no need for physical delivery of the contract.
- Cash settlement can be done only if the contract specifies so.

CONTRACT CREATION

- **Futures contracts have a maximum of 3-month trading cycle - the near month (one), the next month (two) and the far month (three).**
- **New contracts are introduced by the exchange on the trading day following the expiry of the near month contracts.**
- **The new contracts are introduced for a three month duration.**

- **This way, at any point in time, there will be 3 contracts available for trading in the market (for each security) i.e., one near month, one mid month and one far month duration respectively.**

ISSUE OF DEFAULT RISK

- Forwards are private contracts.
- Creditworthiness assessed by participating parties.
- As a consequence, parties are explicitly aware of and accept default risk of counterparty.
- **In futures, due to trading, legs may get transferred to third parties without consent of earlier parties.**
- Therefore, if trading is to be uninhibited, this risk element must be eliminated.
- This risk is eliminated by the exchange clearing house guaranteeing contract performance.